

五南出版

微積分  
初學者的福音

# 白話微積分



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## 本書特色

- ★ 內容深入淺出，適合自學
- ★ 例題解答步驟詳細
- ★ 指出學習盲點

## 試閱本

### 5.9 積分技巧雜談



## ■ 5.9 積分技巧雜談

對於

$$\int_{-3}^3 \sqrt{9-x^2} dx$$

通常想到三角代換。不過積分就是求面積，而被積分函數  $y = \sqrt{3^2 - x^2}$  其實就是上半圓，其半徑為 3，所以直接使用圓面積公式便可得到

$$\int_{-3}^3 \sqrt{9-x^2} dx = \frac{1}{2} \cdot 3^2 \pi$$

除非題目指定使用三角代換，否則遇到這個直接使用圓面積就出來了。

有了這經驗後，我們來看

$$\int_{-2}^2 (x+3) \sqrt{4-x^2} dx$$

先乘開寫成

$$\begin{aligned} & \int_{-2}^2 x \sqrt{4-x^2} + 3 \sqrt{4-x^2} dx \\ &= \int_{-2}^2 x \sqrt{4-x^2} dx + 3 \int_{-2}^2 \sqrt{4-x^2} dx \end{aligned}$$

拆開以後，注意第一項是奇函數的積分，結果為 0；第二項是上半圓的面積再三倍，所以答案就是

$$\int_{-2}^2 (x+3) \sqrt{4-x^2} dx = 0 + 3 \cdot \frac{1}{2} \cdot 2^2 \pi = 6\pi$$

其實應該說一開始就注意到有一塊奇函數的部分，才會想要一開始乘開。現在將這種對稱性的考慮昇華到暴力層次，請看下題。

### 例題 5.9.1

$$\int_{-2}^2 x \ln(1+e^x) dx$$

解

積分範圍讓我們想到是否可以利用奇偶性。首

先分析若  $f(x) = x \ln(1 + e^x)$ , 則  $f(-x) = -x \ln(1 + e^{-x}) = -x \ln\left(\frac{e^x + 1}{e^x}\right) = -x \ln(1 + e^x) + x \ln e^x = -x \ln(1 + e^x) + x^2$ 。差了一點點，實在是壯士扼腕！百般不服氣之下，決定霸王硬上弓，拆成

$$\begin{aligned} & \int_{-2}^2 x \ln(1 + e^x) - \frac{x^2}{2} dx + \int_{-2}^2 \frac{x^2}{2} dx \\ &= 0 + \int_0^2 x^2 dx \end{aligned}$$

其中  $g(x) = x \ln(1 + e^x) - \frac{x^2}{2}$ ,  $g(-x) = -x \ln(1 + e^x) + x^2 - \frac{x^2}{2} = -g(x)$ , 故  $g(x)$  為奇函數。

接下來看一個可以作巧妙互消的積分

$$\int \frac{x e^x}{(x+1)^2} dx$$

先作個  $u = x + 1$  代換讓分母好看一點

$$\begin{aligned} &= \int \frac{(u-1)e^{(u-1)}}{u^2} du \\ &= \frac{1}{e} \cdot \left( \int \frac{e^u}{u} du - \int \frac{e^u}{u^2} du \right) \\ &= \frac{1}{e} \cdot \left( \frac{e^u}{u} + \int \frac{e^u}{u^2} du - \int \frac{e^u}{u^2} du \right) \\ &= \frac{e^{u-1}}{u} + C = \frac{e^x}{x+1} + C \end{aligned}$$

對第一個做分部積分

當遇到這種帶有  $e^x$  的積分

$$\int \frac{e^x}{e^x + 1} dx$$

只要你在學習變數代換技巧時，有培養一定的靈敏度，便知道只要做  $u = e^x$  的代換即可作出。然而如果我將分子改成 1，變成

$$\int \frac{dx}{e^x + 1}$$

那麼就上下同乘以  $e^{-x}$  來強迫湊微分，得到

$$\int \frac{e^{-x} dx}{1 + e^{-x}}$$

接著再設  $u = e^{-x}$  即可。

### 例題 5.9.2

$$\int_{-1}^1 \frac{dx}{(e^x+1)(1+x^2)}$$

解

$$\begin{aligned} & \int_{-1}^0 \frac{dx}{(e^x+1)(1+x^2)} + \int_0^1 \frac{dx}{(e^x+1)(1+x^2)} \\ &= -\int_1^0 \frac{du}{(e^{-u}+1)(1+u^2)} + \int_0^1 \frac{dx}{(e^x+1)(1+x^2)} \\ &= \int_0^1 \frac{e^u du}{(1+e^u)(1+u^2)} + \int_0^1 \frac{dx}{(e^x+1)(1+x^2)} \\ &= \int_0^1 \frac{du}{1+u^2} = \frac{\pi}{4} \end{aligned}$$

### 例題 5.9.3

$$\int \frac{dx}{x + \sqrt{1-x^2}}$$

解 1

看到那個  $1-x^2$ ，比較直覺地會想到三角代換  
 $x = \sin(t)$

$$\begin{aligned} & \int \frac{\cos(t)}{\sin(t) + \cos(t)} dt \\ &= \int \frac{\cos(t)(\sin(t) + \cos(t))}{(\sin(t) + \cos(t))^2} dt \\ &= \frac{1}{2} \int \frac{\sin(2t) + (1 + \cos(2t))}{1 + \sin(2t)} dt \\ &= \frac{1}{2} \int dt + \frac{1}{2} \int \frac{\cos(2t)}{1 + \sin(2t)} dt \end{aligned}$$

$$= \frac{1}{2}t + \frac{1}{4} \ln |1 + \sin(2t)| + C$$

$$= \frac{1}{2} \sin^{-1}(x) + \frac{1}{2} \ln \left| x + \sqrt{1-x^2} \right| + C$$

解 2

$$\int \frac{\cos(t)}{\sin(t) + \cos(t)} dt = \int \frac{\cos(t)(\cos(t) - \sin(t))}{\cos^2(t) - \sin^2(t)} dt$$

$$= \frac{1}{2} \int \frac{(1 + \cos(2t)) - \sin(2t)}{\cos(2t)} dt$$

$$= \frac{1}{2} \int (\sec(2t) + 1 - \tan(2t)) dt$$

$$= \frac{1}{4} (\ln |\sec(2t) + \tan(2t)| + 2t - \ln |\sec(2t)|) + C$$

$$= \frac{1}{2}t + \frac{1}{4} \ln |1 + \sin(2t)| + C$$

解 3

$$\int \frac{\cos(t)}{\sin(t) + \cos(t)} dt = \int \frac{\cos\left[\left(t + \frac{\pi}{4}\right) - \frac{\pi}{4}\right]}{\sqrt{2} \sin\left(t + \frac{\pi}{4}\right)} dt$$

$$= \int \frac{\frac{1}{\sqrt{2}} \cos\left(t + \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}} \sin\left(t + \frac{\pi}{4}\right)}{\sqrt{2} \sin\left(t + \frac{\pi}{4}\right)} dt$$

$$= \frac{1}{2} \int \left(\cot\left(t + \frac{\pi}{4}\right) + 1\right) dt$$

$$= \frac{1}{2}t + \frac{1}{2} \ln \left| \sin\left(t + \frac{\pi}{4}\right) \right| + C$$

$$= \frac{1}{2}t + \frac{1}{2} \ln \left| \frac{1}{\sqrt{2}} \sin(t) + \frac{1}{\sqrt{2}} \cos(t) \right| + C$$

$$\begin{aligned}
 &= \frac{1}{2}t + \frac{1}{2} \ln |\sin(t) + \cos(t)| - \frac{1}{2} \ln \frac{1}{\sqrt{2}} + C \\
 &= \frac{1}{2}t + \frac{1}{4} \ln |1 + \sin(2t)| + C
 \end{aligned}$$

解 4

$$\begin{aligned}
 \int \frac{\cos(t)}{\sin(t) + \cos(t)} dt &= \int \frac{1}{1 + \tan(t)} dt \\
 &= \int \frac{\sec^2(t)}{(1 + \tan(t))(1 + \tan^2(t))} dt \\
 &= \frac{1}{2} \int \left( \frac{1}{1 + \tan(t)} + \frac{1 - \tan(t)}{1 + \tan^2(t)} \right) \sec^2(t) dt \\
 &= \frac{1}{2} \int \left( \frac{1}{1 + \tan(t)} + \frac{1}{1 + \tan^2(t)} - \frac{\tan(t)}{1 + \tan^2(t)} \right) \sec^2(t) dt \\
 &= \frac{1}{2} \int dt + \frac{1}{2} \int \left( \frac{1}{1 + \tan(t)} - \frac{\tan(t)}{1 + \tan^2(t)} \right) \sec^2(t) dt \\
 &= \frac{1}{2}t + \frac{1}{2} \int \frac{du}{1+u} - \frac{1}{2} \int \frac{u du}{1+u^2} = \frac{1}{2}t + \frac{1}{2} \ln |1+u| - \frac{1}{4} \ln |1+u^2| \\
 &= \frac{1}{2}t + \frac{1}{2} \ln |1 + \tan(t)| - \frac{1}{4} \ln |1 + \tan^2(t)| + C \\
 &= \frac{1}{2}t + \frac{1}{4} \ln \left| \frac{(1 + \tan(t))^2}{\sec^2(t)} \right| + C \\
 &= \frac{1}{2}t + \frac{1}{4} \ln \left| 1 + \frac{2 \tan(t)}{\sec^2(t)} \right| + C = \frac{1}{2}t + \frac{1}{4} \ln |1 + \sin(2t)| + C
 \end{aligned}$$

解 5

$$\begin{aligned}
 \int \frac{\cos(t)}{\sin(t) + \cos(t)} dt &= \frac{1}{2} \int \frac{\sin(t) + \cos(t) + \cos(t) - \sin(t)}{\sin(t) + \cos(t)} dt \\
 &= \frac{1}{2} \int dt + \frac{1}{2} \int \frac{\cos(t) - \sin(t)}{\sin(t) + \cos(t)} dt
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \int dt + \frac{1}{2} \ln |\sin(t) + \cos(t)| + C \\
 &= \frac{1}{2} t + \frac{1}{4} \ln |1 + \sin(2t)| + C
 \end{aligned}$$

解 6

三角代換後可用 Weierstrass 代換  $u = \tan(\frac{t}{2})$ ,  $dt = \frac{2 du}{1+u^2}$ ,

$$\sin(t) = \frac{2u}{1+u^2}, \cos(t) = \frac{1-u^2}{1+u^2}$$

$$\begin{aligned}
 \int \frac{\cos(t)}{\sin(t) + \cos(t)} dt &= \int \frac{\frac{1-u^2}{1+u^2}}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \frac{2 du}{1+u^2} \\
 &= \int \frac{2(1-u^2)}{(1+2u-u^2)(1+u^2)} du = \int \left( \frac{1-u}{1+2u-u^2} + \frac{1-u}{1+u^2} \right) du \\
 &= \frac{1}{2} t + \frac{1}{4} \ln |1 + \sin(2t)| + C
 \end{aligned}$$

解 7

直接嘗試  $u = x + \sqrt{1-x^2}$ ,  $du = (1 - \frac{x}{\sqrt{1-x^2}}) dx$ ,  
為了處理多出的  $\frac{x}{\sqrt{1-x^2}}$ , 拆成

$$\begin{aligned}
 &\int \frac{dx}{x + \sqrt{1-x^2}} \\
 &= \int \frac{1 - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}}{x + \sqrt{1-x^2}} dx \\
 &= \int \frac{du}{u} + \int \frac{\frac{x}{\sqrt{1-x^2}}}{x + \sqrt{1-x^2}} dx
 \end{aligned}$$

$$\begin{aligned}
&= \ln|u| + \int \frac{x}{x + \sqrt{1-x^2}} dx \\
&= \ln|u| + \int \frac{dx}{(x + \sqrt{1-x^2})\sqrt{1-x^2}} \\
&= \ln|u| + \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{dx}{x + \sqrt{1-x^2}} \\
&= \ln|u| + \sin^{-1}(x) - \int \frac{dx}{x + \sqrt{1-x^2}} + C
\end{aligned}$$

出現原積分，故由自套法得

$$\int \frac{dx}{x + \sqrt{1-x^2}} = \frac{1}{2} \sin^{-1}(x) + \frac{1}{2} \ln|x + \sqrt{1-x^2}| + C$$

解 8

$$\begin{aligned}
\int \frac{dx}{x + \sqrt{1-x^2}} &= \int \frac{\sqrt{1-x^2}}{(x + \sqrt{1-x^2})\sqrt{1-x^2}} dx \\
&= \frac{1}{2} \int \frac{(x + \sqrt{1-x^2}) - (x - \sqrt{1-x^2})}{(x + \sqrt{1-x^2})\sqrt{1-x^2}} dx \\
&= \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx + \frac{1}{2} \int \frac{1 - \frac{x}{\sqrt{1-x^2}}}{x + \sqrt{1-x^2}} dx \\
&= \frac{1}{2} \sin^{-1}(x) + \frac{1}{2} \ln|x + \sqrt{1-x^2}| + C
\end{aligned}$$

以下介紹更深入的技巧，這些不是一般微積分課程必須知道的，一般讀者可跳過。



## 定理 5.9.1

若  $f(x)$  在  $[a, b]$  上連續，則

$$\begin{aligned}\int_a^b f(x) dx &= \int_a^b f(a+b-x) dx \\ &= \frac{1}{2} \int_a^b (f(x) + f(a+b-x)) dx\end{aligned}$$

證

設  $x = a + b - t$ ,  $dx = -dt$ , 則

$$\begin{aligned}\int_a^b f(x) dx &= - \int_b^a f(a+b-t) dt \\ &= \int_a^b f(a+b-x) dx \\ &= \frac{1}{2} \int_a^b (f(x) + f(a+b-x)) dx\end{aligned}$$

下題便是一個應用。

## 例題 5.9.4

$$\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$$

解 1

常見的做法是先設  $x = \tan(t)$ ,  $dx = \sec^2(t) dt$ , 則

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan(t)) dt$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{4}} \ln\left(\frac{\cos(t) + \sin(t)}{\cos(t)}\right) dt \\
&= \int_0^{\frac{\pi}{4}} \ln(\cos(t) + \sin(t)) dt - \int_0^{\frac{\pi}{4}} \ln(\cos(t)) dt \\
&= \int_0^{\frac{\pi}{4}} \ln\left(\sqrt{2}\cos\left(\frac{\pi}{4} - t\right)\right) dt - \int_0^{\frac{\pi}{4}} \ln(\cos(t)) dt \\
&= \int_0^{\frac{\pi}{4}} \ln\sqrt{2} dt + \int_0^{\frac{\pi}{4}} \ln\left(\cos\left(\frac{\pi}{4} - t\right)\right) dt - \int_0^{\frac{\pi}{4}} \ln(\cos(t)) dt \\
&= \frac{\pi}{4} \ln\sqrt{2} + \int_0^{\frac{\pi}{4}} \ln(\cos(u)) du - \int_0^{\frac{\pi}{4}} \ln(\cos(t)) dt \\
&= \frac{\pi}{8} \ln(2)
\end{aligned}$$

## 解 2

先設  $x = \tan(t)$ ,  $dx = \sec^2(t) dt$ , 則

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \ln(1 + \tan(t)) dt \\
&= \frac{1}{2} \int_0^{\frac{\pi}{4}} \left( \ln(1 + \tan(t)) + \ln\left(1 + \tan\left(\frac{\pi}{4} - t\right)\right) \right) dt \\
&= \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln\left(\left(1 + \tan(t)\right)\left[1 + \tan\left(\frac{\pi}{4} - t\right)\right]\right) dt \\
&= \frac{1}{2} \int_0^{\frac{\pi}{4}} \ln 2 dt = \frac{\pi}{8} \ln(2)
\end{aligned}$$

這個解法用到了上述定理。

另外一個變形是：

## 定理 5.9.2

若  $f(x)$  在  $[a, b]$  上連續, 則

$$\int_a^b f(x) dx = \int_a^{\frac{a+b}{2}} (f(x) + f(a+b-x)) dx = \int_{\frac{a+b}{2}}^b (f(x) + f(a+b-x)) dx$$

證

$$\begin{aligned}
 \int_a^b f(x) dx &= \int_a^{\frac{a+b}{2}} f(x) dx + \int_{\frac{a+b}{2}}^b f(x) dx \\
 &= \int_a^{\frac{a+b}{2}} f(x) dx - \int_{\frac{a+b}{2}}^a f(a+b-t) dt \\
 &= \frac{1}{2} \int_a^{\frac{a+b}{2}} (f(x) + f(a+b-x)) dx
 \end{aligned}$$

## 例題 5.9.5

$$\int_{-\pi}^{\pi} \frac{x \sin^3(x) \tan^{-1}(e^x)}{1 + \cos^2(x)} dx$$

解

此題  $a = -\pi, b = \pi$ , 故有  $\int_{-\pi}^{\pi} f(x) dx = \int_0^{\pi} (f(x) + f(-x)) dx$ 。所以

$$\begin{aligned}
 &\int_{-\pi}^{\pi} \frac{x \sin^3(x) \tan^{-1}(e^x)}{1 + \cos^2(x)} dx \\
 &= \int_0^{\pi} \left( \frac{x \sin^3(x) \tan^{-1}(e^x)}{1 + \cos^2(x)} + \frac{x \sin^3(x) \tan^{-1}(e^{-x})}{1 + \cos^2(x)} \right) dx
 \end{aligned}$$

由於  $\tan^{-1}(u) + \tan^{-1}\left(\frac{1}{u}\right) = \frac{\pi}{2}$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{x \sin^3(x)}{1 + \cos^2(x)} dx$$

又由定理 5.8.1

$$= \frac{\pi}{2} \frac{1}{2} \int_0^{\pi} \left( \frac{x \sin^3(x)}{1 + \cos^2(x)} + \frac{(\pi - x) \sin^3(x)}{1 + \cos^2(x)} \right) dx$$

$$\begin{aligned}
&= \frac{\pi^2}{4} \int_0^\pi \frac{\sin^3(x)}{1 + \cos^2(x)} dx \\
&= \frac{\pi^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^3(x)}{1 + \cos^2(x)} dx \quad \boxed{\int_0^\pi f(\sin\theta) d\theta = 2 \int_0^{\frac{\pi}{2}} f(\sin\theta) d\theta} \\
&= \frac{\pi^2}{2} \int_0^1 \frac{1-u^2}{1+u^2} du = \frac{\pi^2}{2} \int_0^1 \left(-1 + \frac{2}{1+u^2}\right) du \\
&= \frac{\pi^2}{2} \left(2 \tan^{-1}(1) - 1\right) = \frac{\pi^2}{4} (\pi - 2)
\end{aligned}$$

下面是一個特殊的積分技巧，叫做**組合積分法**。找出一個長相相近的積分式，通過解方程式來求積分。

### 例題 5.9.6

$$\int \frac{dx}{(x+1)(x-1)}$$

解

設

$$I = \int \frac{dx}{(x+1)(x-1)}, \quad J = \int \frac{x dx}{(x+1)(x-1)}$$

於是有

$$\begin{cases} I + J = \int \frac{(x+1) dx}{(x+1)(x-1)} = \int \frac{dx}{x-1} = \ln|x-1| + C \\ J - I = \int \frac{(x-1) dx}{(x+1)(x-1)} = \int \frac{dx}{x+1} = \ln|x+1| + C \end{cases}$$

解方程式便可得

$$I = \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C$$

### 例題 5.9.7

$$\int \frac{dx}{(1+x)(1+x^2)}$$

解

設

$$I = \int \frac{dx}{(x+1)(x^2+1)}, \quad J = \int \frac{x^2 dx}{(x+1)(x^2+1)}$$

於是有

$$\begin{cases} I+J = \int \frac{(x^2+1) dx}{(x+1)(x^2+1)} = \int \frac{dx}{1+x} = \ln|1+x| + C \\ J-I = \int \frac{(x^2-1) dx}{(x+1)(x^2+1)} = \int \frac{(x-1) dx}{x^2+1} = \frac{1}{2} \ln|x^2+1| - \tan^{-1}(x) \end{cases}$$

解方程式便可得

$$I = \frac{1}{2} \left[ \ln|x+1| - \frac{1}{2} \ln|x^2+1| + \tan^{-1}(x) \right] + C$$

### 例題 5.9.8

$$\int \frac{7 \cos(x) - 3 \sin(x)}{5 \cos(x) + 2 \sin(x)} dx$$

解

設

$$I = \int \frac{\sin(x)}{5 \cos(x) + 2 \sin(x)} dx, \quad J = \int \frac{\cos(x)}{5 \cos(x) + 2 \sin(x)} dx$$

於是有

$$\begin{cases} 2I + 5J = \int dx = x + C \\ 2J - 5I = \int \frac{2 \cos(x) - 5 \sin(x)}{5 \cos(x) + 2 \sin(x)} dx = \ln|5 \cos(x) + 2 \sin(x)| + C \end{cases}$$

解方程式便可得

$$\begin{cases} I = \frac{1}{29} \left[ 2x - 5 \ln|5 \cos(x) + 2 \sin(x)| \right] + C \\ J = \frac{1}{29} \left[ 5x + 2 \ln|5 \cos(x) + 2 \sin(x)| \right] + C \end{cases}$$

則所求為

$$7J - 3I = x + \ln|5 \cos(x) + 2 \sin(x)| + C$$

以下是一些令人意想不到的代換。

## 例題 5.9.9

$$\int \frac{dx}{x^2(a+bx)^2}$$

解

設  $u = \frac{a+bx}{x}$ ,  $x = \frac{a}{u-b}$ ,  $dx = \frac{-a du}{(u-b)^2}$ ,  $a+bx = xu = \frac{au}{u-b}$ , 於是代換成

$$-\frac{1}{a^3} \int \frac{(u-b)^2 du}{u^2}$$

## 例題 5.9.10

$$\int \frac{dx}{\sqrt[3]{(x-1)(x+1)^2}}$$

解

先變形為

$$\int \sqrt[3]{\frac{x+1}{x-1}} \frac{dx}{x+1}$$

接著設  $u = \sqrt[3]{\frac{x+1}{x-1}}$ , 最後便可化為

$$-3 \int \frac{du}{u^3 - 1}$$

再用部分分式求出。

## 例題 5.9.11

$$\int \frac{x dx}{(1+x^3)^{\frac{2}{3}}}$$

解

設  $1+x^3 = \frac{1}{1-t^3}$ ,  $3x^2 dx = \frac{3t^2 dt}{(1-t^3)^2}$ , 便可代換成

$$\int \frac{t dt}{1-t^3}$$

再用部分分式求出。